VI. *Addition to the Memoir on* Tschirnhausen's *Transformation.*

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In the memoir "On Tschirnhausen's Transformation," Philosophical Transactions, vol. clii. (1862) pp. 561–568, I considered the case of a quartic equation: viz. it was shown that the equation

$$(a, b, c, d, e \between x, 1)^4 = 0$$

is, by the substitution

$$y = (ax+b)\mathrm{B} + (ax^2+4bx+3c)\mathrm{C} + (ax^3+4bx^2+6cx+3d)\mathrm{D},$$

transformed into

$$(1, 0, \mathfrak{C}, \mathfrak{D}, \mathfrak{E} \between y, 1)^4 = 0,$$

where $(\mathfrak{C}, \mathfrak{D}, \mathfrak{E})$ have certain given values. It was further remarked that $(\mathfrak{C}, \mathfrak{D}, \mathfrak{E})$ were expressible in terms of $\mathrm{U}', \mathrm{H}', \Phi'$, invariants of the two forms $(a, b, c, d, e \between \mathrm{X}, \mathrm{Y})^4$, $(\mathrm{B}, \mathrm{C}, \mathrm{D} \between \mathrm{Y}, -\mathrm{X})^2$, of I, J, the invariants of the first, and of Θ', $=\mathrm{BD}-\mathrm{C}^2$, the invariant of the second of these two forms,—viz. that we have

$$\mathfrak{C} = 6\mathrm{H}' - 2\mathrm{I}\Theta',$$

$$\mathfrak{D} = 4\Phi',$$

$$\mathfrak{E} = \mathrm{IU}'^3 - 3\mathrm{H}'^2 + \mathrm{I}^2\Theta'^2 + 12\mathrm{J}'\Theta'\mathrm{U}' + 2\mathrm{I}'\Theta'\mathrm{H}'.$$

And by means of these I obtained an expression for the quadrinvariant of the form

$$(1, 0, \mathfrak{C}, \mathfrak{D}, \mathfrak{E} \between y, 1)^4;$$

viz. this was found to be

$$= \mathrm{IU}'^2 + \tfrac{4}{3}\mathrm{I}^2\Theta'^2 + 12\mathrm{J}\Theta'\mathrm{U}'.$$

But I did not obtain an expression for the cubinvariant of the same function: such expression, it was remarked, would contain the square of the invariant Φ'; it was probable that there existed an identical equation,

$$\mathrm{JU}'^3 - \mathrm{IU}'^2\mathrm{H}' + 4\mathrm{H}'^3 + \mathrm{M}\Theta' = -\Phi'^2,$$

which would serve to express Φ'^2 in terms of the other invariants; but, assuming that such an equation existed, the form of the factor M remained to be ascertained; and until this was done, the expression for the cubinvariant could not be obtained in its most simple form. I have recently verified the existence of the identical equation just referred to, and have obtained the expression for the factor M; and with the assistance of this identical equation I have obtained the expression for the cubinvariant of the form

$$(1, 0, \mathfrak{C}, \mathfrak{D}, \mathfrak{E} \between y, 1)^4.$$

The expression for the quadrinvariant was, as already mentioned, given in the former memoir: I find that the two invariants are in fact the invariants of a certain linear function of U, H; viz. the linear function is $=U'U+\frac{2}{3}\Theta'H$; so that, denoting by I*, J*, the quadrinvariant and the cubinvariant respectively of the form

$$(1,\ 0,\ \mathfrak{C},\ \mathfrak{D},\ \mathfrak{E}\between y,\ 1)^4,$$

we have

$$I^*=\tilde{I}(U'U+4\Theta'H),$$

$$J^*=\tilde{J}(U'U+4\Theta'H),$$

where $\tilde{I}$, $\tilde{J}$ signify the functional operations of forming the two invariants respectively. The function $(1,\ 0,\ \mathfrak{C},\ \mathfrak{D},\ \mathfrak{E}\between y,\ 1)^4$, obtained by the application of TSCHIRNHAUSEN'S transformation to the equation

$$(a,\ b,\ c,\ d,\ e\between x,\ 1)^4=0,$$

has thus the *same invariants* with the function

$$U'U+4\Theta'H=U'(a,\ b,\ c,\ d,\ e\between x,\ 1)^4+4\Theta'(ac-b^2,\ ad-bc,\ ae+2bd-3c^2,\ be-cd,\ ce-d^2\between x,\ 1)^4,$$

and it is consequently a linear transformation of the last-mentioned function; so that the application of TSCHIRNHAUSEN'S transformation to the equation $U=0$ gives an equation linearly transformable into, and thus virtually equivalent to, the equation

$$U'U+4\Theta'H=0,$$

which is an equation involving the single parameter $\frac{4\Theta'}{U'}$: this appears to me a result of considerable interest. It is to be remarked that TSCHIRNHAUSEN'S transformation, wherein y is put equal to a rational and integral function of the order $n-1$ (if n be the order of the equation in x), is not really less general than the transformation wherein y is put equal to any rational function $\frac{V}{W}$ whatever of x; such rational function may, in fact, by means of the given equation in x, be reduced to a rational and integral function of the order $n-1$; hence in the present case, taking V, W to be respectively of the order $n-1$, $=3$, it follows that the equation in y obtained by the elimination of x from the equations

$$(a,\ b,\ c,\ d,\ e\between x,\ 1)^4=0,$$

$$y=\frac{(\alpha,\ \beta,\ \gamma,\ \delta\between x,\ 1)^3}{(\alpha',\ \beta',\ \gamma',\ \delta'\between x,\ 1)^3}$$

is a mere linear transformation of the equation $AU+BH=0$, where A, B are functions (not as yet calculated) of $(a,\ b,\ c,\ d,\ e,\ \alpha,\ \beta,\ \gamma,\ \delta,\ \alpha',\ \beta',\ \gamma',\ \delta')$.

Article Nos. 1, 2, 3.—*Investigation of the identical equation*

$$JU'^3-IU'^2H'+4H'^3+M\Theta'=-\Phi'^2.$$

1. It is only necessary to show that we have such an equation, M being an invariant,

in the particular case $a=e=1$, $b=d=0$, $c=\theta$, that is for the quartic function $(1, 0, \theta, 0, 1)(x, 1)^4$; for, this being so, the equation will be true in general. Writing the equation in the form

$$-\mathrm{M}\Theta'=\mathrm{U}'^2(\mathrm{JU}'-\mathrm{IH}')+4\mathrm{H}'^3+\Phi'^2,$$

and observing that we have

$$\begin{aligned}
\mathrm{U}'&=(\mathrm{B}^2+\mathrm{D}^2)+2\theta\mathrm{BD}+4\theta\mathrm{C}^2,\\
\mathrm{H}'&=\theta(\mathrm{B}^2+\mathrm{D}^2)+(1+\theta^2)\mathrm{BD}-4\theta^2\mathrm{C}^2,\\
\Theta'&=\mathrm{BD}-\mathrm{C}^2,\\
\Phi'&=(1-9\theta^2)\mathrm{C}(\mathrm{B}^2-\mathrm{D}^2),\\
\mathrm{I}&=1+3\theta^2,\\
\mathrm{J}&=\theta-\theta^3,
\end{aligned}$$

and thence

$$\mathrm{JU}'-\mathrm{IH}'=-4\theta^3(\mathrm{B}^2+\mathrm{D}^2)+(-1-2\theta^2-5\theta^4)\mathrm{BD}+(8\theta^2+8\theta^4)\mathrm{C}^2,$$

the equation becomes

$$\begin{aligned}
-(\mathrm{BD}-\mathrm{C}^2)\mathrm{M}= &\\
&\{-4\theta^3(\mathrm{B}^2+\mathrm{D}^2)+(-1-2\theta^2-5\theta^4)\mathrm{BD}+(8\theta^2+8\theta^4)\mathrm{C}^2\}\\
&\qquad\times\{\mathrm{B}^2+\mathrm{D}^2+2\theta\mathrm{BD}+4\theta\mathrm{C}^2\}^2\\
&+4\{\theta(\mathrm{B}^2+\mathrm{D}^2)+(1+\theta^2)\mathrm{BD}-4\theta^2\mathrm{C}^2\}^3\\
&+(1-9\theta^2)^2\,\mathrm{C}^2\{(\mathrm{B}^2+\mathrm{D}^2)^2-4\mathrm{B}^2\mathrm{D}^2\}.
\end{aligned}$$

2. It is found by developing that the right-hand side is in fact divisible by $\mathrm{BD}-\mathrm{C}^2$, and that the quotient is

$$\begin{aligned}
=\ &(-1+10\theta^2-9\theta^4)(\mathrm{B}^2+\mathrm{D}^2)^2\\
&+(8\theta+16\theta^3-24\theta^5)(\mathrm{B}^2+\mathrm{D}^2)\mathrm{BD}\\
&+(4+8\theta^2+4\theta^4-16\theta^6)\mathrm{B}^2\mathrm{D}^2\\
&+(-64\theta^3-192\theta^5)(\mathrm{B}^2+\mathrm{D}^2)\mathrm{C}^2\\
&+(16\theta^2-416\theta^4-112\theta^6)\mathrm{BDC}^2\\
&+(-128\theta^4+128\theta^6)\mathrm{C}^4.
\end{aligned}$$

3. This is found to be

$$\begin{aligned}
=&-\mathrm{I}^2\mathrm{U}'^2+12\mathrm{JU}'\mathrm{H}'+4\mathrm{IH}'^2\\
&-\ 8\mathrm{IJU}'\Theta'\\
&-16\mathrm{J}^2\Theta'^2,
\end{aligned}$$

which is consequently the value of $-\mathrm{M}$. We have therefore

$$\begin{aligned}
-\Phi'^2=\ &\mathrm{JU}'^3-\mathrm{IU}'^2\mathrm{H}'+4\mathrm{H}'^3\\
&+(\mathrm{I}^2\mathrm{U}'^2-12\mathrm{JU}'\mathrm{H}'-4\mathrm{IH}'^2)\Theta'\\
&+\ 8\mathrm{IJU}'\Theta'^2\\
&+16\mathrm{J}^2\Theta'^3,
\end{aligned}$$

which is the required identical equation.

Article No. 4.—*Calculation of the Cubinvariant.*

4. We have

$$\begin{aligned} J^* &= \tfrac{1}{6}\mathfrak{C}\,.\,\mathfrak{E}-(\tfrac{1}{6}\mathfrak{C})^3-(\tfrac{1}{4}\mathfrak{D})^2 \\ &= (H-\tfrac{1}{3}I\Theta')\{IU'^2-3H'^2+(12JU'+2IH')\Theta'+I^2\Theta'^2\} \\ &\quad -(H-\tfrac{1}{3}I\Theta')^3 \\ &\quad -\Phi'^2, \end{aligned}$$

whence, substituting for $-\Phi'^2$ its value and reducing, we find

$$J^*=JU'^3+\Theta'.\tfrac{2}{3}I^2U'^2+\Theta'^2(4IJU')+\Theta'^3(16J^2-\tfrac{8}{27}I^3).$$

Article No. 5.—*Final expressions of the two Invariants.*

The value of I* has been already mentioned to be $I^*=IU'^2+\Theta'12JU'+\Theta'^2.\tfrac{4}{3}I^2$, and it hence appears that the values of the two invariants may be written

$$\begin{aligned} I^* &= (I,\ 18J,\ 3I^2\between U',\ \tfrac{2}{3}\Theta')^2, \\ J^* &= (J,\ I^2,\ 9IJ,\ -I^3+54J^2\between U',\ \tfrac{2}{3}\Theta')^3. \end{aligned}$$

But we have (see Table No. 72 in my "Seventh Memoir on Quantics" †)

$$\begin{aligned} \tilde{I}(\alpha U+6\beta H) &= (I,\ 18J,\ 3I^2\between\alpha,\ \beta)^2 \\ \tilde{J}(\alpha U+6\beta H) &= (J,\ I^2,\ 9IJ,\ -I^3+54J^2\between\alpha,\ \beta)^3; \end{aligned}$$

so that, writing $\alpha=U'$, $\beta=\tfrac{2}{3}\Theta'$, we have

$$\begin{aligned} I^* &= \tilde{I}(U'U+4\Theta'H), \\ J^* &= \tilde{J}(U'U+4\Theta'H); \end{aligned}$$

or the function $(1,\ 0,\ \mathfrak{C},\ \mathfrak{D},\ \mathfrak{E}\between y,\ 1)^4$ obtained from TSCHIRNHAUSEN'S transformation of the equation $U=0$ has the same invariants with the function $U'U+4\Theta'H$; or, what is the same thing, the equation $(1,\ 0,\ \mathfrak{C},\ \mathfrak{D},\ \mathfrak{E}\between y,\ 1^4)=0$ is a mere linear transformation of the equation $U'U+4\Theta H=0$; which is the above-mentioned theorem.

† Philosophical Transactions, vol. cli. (1861), pp. 277–292.